Einführung in die Theoretische Physik

Blatt 9

Aufgabe 32

a) rechter Seite eingesetzt: $\int_{-\infty}^{+\infty} dx\, f(x) x \delta(x) = f(x) x|_{x=0} = 0$

linke Seite: $\int_{-\infty}^{+\infty} dx\, f(x) 0 = 0$

b) rechter Seite eingesetzt: $\int_{-\infty}^{+\infty} dx\, f(x)\varphi(x)\delta(x-a) = f(x)\varphi(x)|_{x=a} = f(a)\varphi(a)$

linke Seite: $\int_{-\infty}^{+\infty} dx\, f(x)\varphi(a)\delta(x-a) = f(x)\varphi(a)|_{x=a} = f(a)\varphi(a)$

c) rechter Seite eingesetzt: $\int_{-\infty}^{+\infty} dx\, f(x) \int_{-\infty}^{+\infty} dy\, \delta(x-y)\delta(y-z) =$

$$= \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\, f(x)\, \delta(x-y)\delta(y-z) = \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx\, f(x)\, \delta(x-y)\delta(y-z) =$$

$$= \int_{-\infty}^{+\infty} dy\, f(y)\delta(y-z) = f(z)$$

linke Seite: $\int_{-\infty}^{+\infty} dx\, f(x)\delta(x-z) = f(z)$

d) z.z.: $\delta^{(m)}(x) = (-1)^m \delta^{(m)}(-x)$

laut Angabe: $\int_{-\infty}^{+\infty} dy\, f(y)\delta^{(m)}(y-x) = (-1)^m \frac{d^m f(x)}{dx^m}$ (*)

linke Seite: $\int_{-\infty}^{+\infty} dy\, (-1)^m f(y)\delta^{(m)}(x-y) = \int_{-\infty}^{+\infty} dy\, (-1)^m f(-y)\delta^{(m)}(y-x) \overset{\text{nach } (*)}{=}$

$$= (-1)^m \frac{d^m f(-x)}{dx^m} \underset{\text{für } f(-x)}{\overset{m\text{-mal Kettenr.}}{=}} (-1)^m (-1)^m \frac{d^m f(x)}{dx^m} = \frac{d^m f(x)}{dx^m}$$

Aufgabe 33

Kugelschale: $\int_K \rho dV = \int_{r_1}^{r_2} dr \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, r^2 \sin\theta\, \rho = 4\pi \int_{r_1}^{r_2} dr\, r^2 \rho \overset{!}{=} \begin{cases} 0 & R \in [r_1, r_2] \\ q & R \notin [r_1, r_2] \end{cases} \Rightarrow \rho = \frac{q}{4\pi R^2}\delta(\|\vec{x}\| - R)$

Kreisscheibe: $\rho = c\, q\, \delta(z)\, \Theta(R - \sqrt{x^2+y^2})$

$$\int_{\mathbb{R}^3} \rho d^3\vec{r} = \int_0^{\infty} dr \int_0^{2\pi} d\varphi \int_{-\infty}^{+\infty} dz\, r\, c\, q\, \delta(z)\, \Theta(R-r) = 2\pi \int_0^{\infty} dr \int_{-\infty}^{+\infty} dz\, r\, c\, q\, \delta(z)\, \Theta(R-r) =$$

$$= 2\pi R^2 c\, q \Rightarrow c = \frac{1}{2\pi R^2}$$

$$\Rightarrow \rho = \frac{1}{2\pi R^2} q\, \delta(z)\, \Theta(R - \sqrt{x^2+y^2})$$

Aufgabe 34

zu überprüfen: $\int\limits_{\partial K} df \cdot \vec{n} \cdot \vec{E} = 4\pi \int\limits_K \rho d^3 r = 4\pi q$

$\vec{E}(\vec{r}) = q \frac{\vec{r} - \vec{a}}{|\vec{r} - \vec{a}|^3}$

$x = r\cos\varphi\cos\theta$

$y = r\sin\varphi\cos\theta$

$z = r\sin\theta$

w.l.o.g.: $\vec{a} = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ a'R \end{pmatrix}$

Tangenten Vektoren an Kugeloberfläche: $\vec{t}_\varphi$ and $\vec{t}_\theta$

$$\vec{t}_\varphi = \frac{\partial \vec{r}}{\partial \varphi} = \begin{pmatrix} -r\cos\theta\sin\varphi \\ r\cos\theta\cos\varphi \\ 0 \end{pmatrix}$$

$$\vec{t}_\theta = \frac{\partial \vec{r}}{\partial \theta} = \begin{pmatrix} -r\cos\varphi\sin\theta \\ -r\sin\varphi\sin\theta \\ r\cos\theta \end{pmatrix}$$

$$\Rightarrow \vec{n} = \vec{t}_\varphi \times \vec{t}_\theta = r^2 \begin{pmatrix} \cos^2\theta\cos\varphi \\ \cos^2\theta\sin\varphi \\ \sin^2\varphi\sin\theta\cos\theta + \cos^2\varphi\cos\theta\sin\theta \end{pmatrix} = r^2 \begin{pmatrix} \cos^2\theta\cos\varphi \\ \cos^2\theta\sin\varphi \\ \sin\theta\cos\theta \end{pmatrix}$$

$$\int\limits_{\partial K} df \cdot \vec{n} \cdot \vec{E} = \int\limits_0^{2\pi} d\varphi \int\limits_{-\pi/2}^{\pi/2} d\theta\, R^2 \begin{pmatrix} \cos^2\theta\cos\varphi \\ \cos^2\theta\sin\varphi \\ \sin\theta\cos\theta \end{pmatrix} \cdot q \frac{\vec{r} - \vec{a}}{|\vec{r} - \vec{a}|^3} =$$

$$= \int\limits_0^{2\pi} d\varphi \int\limits_{-\pi/2}^{\pi/2} d\theta\, R^2 \begin{pmatrix} \cos^2\theta\cos\varphi \\ \cos^2\theta\sin\varphi \\ \sin\theta\cos\theta \end{pmatrix} \cdot q \frac{R\begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ \sin\theta - a' \end{pmatrix}}{\left|R\begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ \sin\theta - a' \end{pmatrix}\right|^3} = q\int\limits_0^{2\pi} d\varphi \int\limits_{-\pi/2}^{\pi/2} d\theta \begin{pmatrix} \cos^2\theta\cos\varphi \\ \cos^2\theta\sin\varphi \\ \sin\theta\cos\theta \end{pmatrix} \frac{\begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ \sin\theta - a' \end{pmatrix}}{\left|\begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ \sin\theta - a' \end{pmatrix}\right|^3} =$$

$$= q\int\limits_0^{2\pi} d\varphi \int\limits_{-\pi/2}^{\pi/2} d\theta\, \cos\theta \frac{(\cos^2\theta\cos^2\varphi + \cos^2\theta\sin^2\varphi + \sin^2\theta - \sin\theta\, a)}{|(\cos^2\theta\cos^2\varphi + \cos^2\theta\sin^2\varphi + \sin^2\theta - 2\sin\theta\, a + a^2)\begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ \sin\theta - a \end{pmatrix}|} =$$

$$= q\int\limits_0^{2\pi} d\varphi \int\limits_{-\pi/2}^{\pi/2} d\theta\, \cos\theta \frac{\cos^2\theta + \sin^2\theta - \sin\theta\, a'}{|(\cos^2\theta + \sin^2\theta - 2\sin\theta\, a' + a'^2)\begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ \sin\theta - a' \end{pmatrix}|} =$$

$$= q\int\limits_0^{2\pi} d\varphi \int\limits_{-\pi/2}^{\pi/2} d\theta\, \frac{\cos\theta(1 - a'\sin\theta)}{\left|\begin{pmatrix} \cos\varphi\cos\theta(1 - 2a'\sin\theta + a'^2) \\ \sin\varphi\cos\theta(1 - 2a'\sin\theta + a'^2) \\ (\sin\theta - a')(1 - 2a'\sin\theta + a'^2) \end{pmatrix}\right|} = q\int\limits_0^{2\pi} d\varphi \int\limits_{-\pi/2}^{\pi/2} d\theta\, \frac{\cos\theta(1 - a'\sin\theta)}{(1 - 2a'\sin\theta + a'^2)^{\frac{3}{2}}} =$$

$$2\pi q \int\limits_{-\pi/2}^{\pi/2} d\theta\, \frac{\cos\theta(1 - a'\sin\theta)}{(1 - 2a'\sin\theta + a'^2)^{\frac{3}{2}}} = 2\pi q \int\limits_{-1}^{1} db\, \frac{1 - a'b}{(1 - 2a'b + a'^2)^{\frac{3}{2}}} =$$

$$= 2\pi q \int\limits_{-1}^{1} db\, \frac{1}{(1 - 2a'b + a'^2)^{\frac{3}{2}}} - \frac{a'b}{(1 - 2a'b + a'^2)^{\frac{3}{2}}} = 2\pi q \left(\left[\frac{1}{a'\sqrt{1 - 2a'b + a'^2}} \right]_{-1}^{1} - \int\limits_{-1}^{1} db\, \frac{a'b}{(1 - 2a'b + a'^2)^{\frac{3}{2}}} \right) =$$

$$= 2\pi q \left(\frac{1}{a'\sqrt{(1 - a')^2}} - \frac{1}{a'\sqrt{(1 + a')^2}} - \left[\frac{b}{\sqrt{1 - 2a'b + a^2}} \right]_{-1}^{1} + \int\limits_{-1}^{1} db \frac{1}{\sqrt{1 - 2a'b + a^2}} \right) =$$

$$= 2\pi q \left(\frac{1}{a'(1 - a')} - \frac{1}{a'(1 + a')} - \frac{1}{\sqrt{(1 - a')^2}} - \frac{1}{\sqrt{(1 + a')^2}} + \left[\frac{\sqrt{1 - 2a'b + a'^2}}{-a'} \right]_{-1}^{1} \right) =$$

$$= 2\pi q \left(\frac{1}{a'(1 - a')} - \frac{1}{a'(1 + a')} - \frac{1}{(1 - a')} - \frac{1}{(1 + a')} - \frac{\sqrt{(1 - a')^2}}{a'} + \frac{\sqrt{(1 + a')^2}}{a'} \right) =$$

$$= 2\pi q \left(\frac{1 + a' - 1 + a' - a' - a'^2 - a' + a'^2 - (1 - a')^2(1 + a') + (1 - a')(1 + a')^2}{a'(1 - a')(1 + a')} \right) =$$

$$= 2\pi q\left(\frac{-(1-a')(1+a')+a'(1-a')(1+a')+(1-a')(1+a')+a'(1-a')(1+a')}{a'(1-a')(1+a')}\right) = 2\pi q\left(\frac{2a'(1-a')(1+a')}{a'(1-a')(1+a')}\right) =$$
$$= 4\pi q$$

Aufgabe 36 (oder 35??? ;-))

a) $\phi(\vec{r}) = \int d^3r' \frac{\varrho(\vec{r'})}{|\vec{r}-\vec{r'}|}$

$$\varrho(\vec{r}) = \begin{cases} \frac{q}{\frac{4}{3}\pi R^3} & \text{for } |\vec{r}\,| \leq R \\ 0 & \text{for } |\vec{r}\,| > R \end{cases}$$

$$\vec{r'} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} r'\cos\varphi\cos\vartheta \\ r'\sin\varphi\cos\vartheta \\ r'\sin\vartheta \end{pmatrix}$$

here: $\vec{r} = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}$

$$\phi(\vec{r}) = \int d^3\vec{r'} \frac{\varrho(\vec{r'})}{|\vec{r}-\vec{r'}|} = \int_0^R dr' \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\vartheta r'^2 \cos\vartheta \frac{q}{\frac{4}{3}\pi R^3} \left| \begin{pmatrix} r'\cos\varphi\cos\vartheta \\ r'\sin\varphi\cos\vartheta \\ r'\sin\vartheta - r \end{pmatrix} \right|^{-1} =$$

$$= \frac{q}{\frac{4}{3}\pi R^3} \int_0^R dr' \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\vartheta r'^2 \cos\vartheta \frac{1}{\sqrt{r'^2\cos^2\varphi\cos^2\vartheta + r'^2\sin^2\varphi\cos^2\vartheta + r'^2\sin^2\vartheta - 2r\,r'\sin\vartheta + r^2}} =$$

$$= \frac{q}{\frac{4}{3}\pi R^3} \int_0^R d\,r' \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\vartheta \frac{r'^2\cos\vartheta}{\sqrt{r'^2\cos^2\vartheta + r'^2\sin^2\vartheta - 2r\,r'\sin\vartheta + r^2}} = \frac{q\cdot 2\pi}{\frac{4}{3}\pi R^3} \int_0^R d\,r' \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\vartheta \frac{r'^2\cos\vartheta}{\sqrt{r'^2 - 2r\,r'\sin\vartheta + r^2}} =$$

$\alpha := \sin\vartheta$
$d\alpha = \cos\vartheta d\vartheta$

$$= \frac{q}{\frac{2}{3}R^3} \int_0^R d\ r' \int_{-1}^{+1} d\alpha \frac{r'^2}{\sqrt{(r'-r)^2 + 2r\,r' - 2r\,r'\alpha}} = \frac{3q}{2R^3} \int_0^R d\ r' \int_{-1}^{+1} d\alpha \frac{r'^2}{\sqrt{(r'-r)^2 + 2r\,r' - 2r\,r'\alpha}} = \quad \text{with } \int \frac{d\,x}{\sqrt{a\,x+b}} = \frac{2\sqrt{a\,x+b}}{a}$$

$$= \frac{3q}{2R^3} \int_0^R d\,r' \left[\frac{2r'^2\sqrt{(r'-r)^2 + 2r\,r' - 2r\,r'\alpha}}{-2r\,r'}\right]_{-1}^{1} = \frac{3q}{2R^3} \int_0^R d\,r' \frac{r'\sqrt{(r'-r)^2} - r'\sqrt{(r'-r)^2 + 4r\,r'}}{-r} =$$

$$= \frac{3q}{2R^3} \int_0^R d\,r' \frac{r'\sqrt{(r'+r)^2} - r'\sqrt{(r'-r)^2}}{r}$$

for $|\vec{r}\,| = r < R$:

$$\phi(\vec{r}) = \frac{3q}{2R^3 r}\left(\int_0^r d\,r'(r'\sqrt{(r'+r)^2} - r'\sqrt{(r'-r)^2}) + \int_r^R d\,r'(r'\sqrt{(r'+r)^2} - r'\sqrt{(r'-r)^2})\right) =$$

$$= \frac{3q}{2R^3 r}\left(\int_0^r d\,r'(r'(r'+r) + r'(r'-r)) + \int_r^R d\,r'(r'(r'+r) - r'(r'-r))\right) = \frac{3q}{2R^3 r}\left(\int_0^r d\,r'\,2r'^2 + \int_r^R d\,r'\,2r'r\right) =$$

$$\frac{3q}{2R^3 r}\left(\left[\frac{2}{3}r'^3\right]_0^r + r\left[r'^2\right]_r^R\right) = \frac{3q}{2R^3 r}\left(\frac{2}{3}r^3 + r\,R^2 - r^3\right) = \frac{3q}{2R^3 r}\left(r\,R^2 - \frac{1}{3}r^3\right) = \frac{3q}{2R} - \frac{q\,r^2}{2R^3}$$

$$\Rightarrow \vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r}) = -\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}\left(\frac{3q}{2R} - \frac{q\,r^2}{2R^3}\right) = -\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}\left(-\frac{q\,(x^2+y^2+z^2)}{2R^3}\right) = \frac{q}{2R^3}\begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = \frac{q}{R^3}\vec{r}$$

for $|\vec{r}\,| = r \geq R$:

$$\phi(\vec{r}) = \frac{3q}{2R^3 r}\int_0^R d\, r'(r'\sqrt{(r'+r)^2} - r'\sqrt{(r'-r)^2}) = \frac{3q}{2R^3 r}\int_0^R d\, r'(r'(r' + r) + r'(r' - r)) = \frac{3q}{2R^3 r}\int_0^R d\, r'\, 2r'^2 =$$

$$= \frac{3q}{2R^3 r}\left[\frac{2}{3}r'^3\right]_0^R = \frac{q}{r}$$

$$\Rightarrow \vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r}) = -\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}(\tfrac{q}{r}) = -\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}\left(\frac{q}{\sqrt{x^2+y^2+z^2}}\right) = -q\begin{pmatrix} \frac{-1\cdot 2x}{2\sqrt{x^2+y^2+z^2}^3} \\ \frac{-1\cdot 2y}{2\sqrt{x^2+y^2+z^2}^3} \\ \frac{-1\cdot 2z}{2\sqrt{x^2+y^2+z^2}^3} \end{pmatrix} = \frac{q}{|\vec{r}|^3}\vec{r}$$

b) using cylindrical coordinates $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t\cos\varphi \\ t\sin\varphi \\ z \end{pmatrix}$

$$\rho(\vec{r}) = \begin{cases} \frac{q}{\pi R^2 L} & \text{for } t \le R,\ |z| \le \frac{L}{2} \\ 0 & \text{else} \end{cases}$$

$\leftarrow$ better would be $\rho = \frac{q}{\pi R^2}$ independent of L, otherwise problem for $L \to \infty$ $(\rho = 0)$

Gauß $\Rightarrow \oint_A \vec{E}\, d\vec{A} = \int_V \vec{\nabla}\vec{E}\, dV = \int_V 4\pi\rho\, dV$

for $L \to \infty$: $\vec{E} \perp d\vec{A}$ and $E = \text{const.}$ on any cylinder surface around $z-$axis

$\Rightarrow \oint_A \vec{E}\, d\vec{A} = E \cdot 2\pi\, t\, L$

inside the cylinder:

$$\int_V 4\pi\rho\, dV = 4\pi\int_0^t d\, t' \int_0^{2\pi} d\varphi \int_{-\frac{L}{2}}^{+\frac{L}{2}} d\, z\, t' \frac{q}{\pi R^2 L} = 4\pi\frac{q}{\pi R^2}2\pi\left[\frac{t'^2}{2}\right]_0^t = 4\pi q\frac{t^2}{R^2}$$

$$\Rightarrow E = \frac{4\pi q\frac{t^2}{R^2}}{2\pi\, t\, L} = 2\frac{q}{L}\frac{t}{R^2} \Rightarrow \vec{E}(\vec{r}) = \frac{2q}{L}\frac{\sqrt{x^2+y^2}}{R^2}\vec{e_t}$$

outside the cylinder:

$$\int_V 4\pi\rho\, dV = 4\pi\int_0^R d\, t' \int_0^{2\pi} d\varphi \int_{-\frac{L}{2}}^{+\frac{L}{2}} d\, z\, t' \frac{q}{\pi R^2 L} = 4\pi\frac{q}{\pi R^2}2\pi\left[\frac{t'^2}{2}\right]_0^R = 4\pi q$$

$$\Rightarrow E = \frac{4\pi q}{2\pi\, t\, L} = 2\frac{q}{L}\frac{1}{t} \Rightarrow \vec{E}(\vec{r}) = \frac{2q}{L}\frac{1}{\sqrt{x^2+y^2}}\vec{e_t}$$