Einführung in die Theoretische Physik

Blatt 8

Aufgabe 28

$$g(x)=\int_{-\infty}^{+\infty} dk\, e^{\frac{-l^2k^2}{2}}e^{ikx}=\int_{-\infty}^{+\infty} dk\, e^{\frac{-l^2k^2}{2}+ikx}=\int_{-\infty}^{+\infty} dk\, e^{-\frac{l^2}{2}(k-\frac{ix}{l^2})^2-\frac{x^2}{2l^2}}=\sqrt{\frac{2\pi}{l^2}}e^{-\frac{x^2}{2l^2}}=\frac{d_l(x)}{2\pi}$$

$$\Rightarrow \delta(x)=\lim_{l\to 0}\int_{-\infty}^{+\infty} dk\frac{1}{2\pi}e^{\frac{-l^2k^2}{2}}e^{ikx}=\int_{-\infty}^{+\infty} dk\frac{1}{2\pi}e^{ikx}$$

Aufgabe 29

$$\frac{1}{2}\sum_{n=-N}^{N} e^{i\pi n x}=\frac{1}{2}(\sum_{n=0}^{N}(e^{i\pi nx}+e^{-i\pi nx})-1)=\frac{1}{2}(\sum_{n=0}^{N}\frac{(e^{\frac{i\pi x}{2}}-e^{-\frac{i\pi x}{2}})(e^{i\pi nx}+e^{-i\pi nx}))}{e^{\frac{i\pi x}{2}}-e^{-\frac{i\pi x}{2}}}-1)=$$

$$=\frac{1}{2}(\sum_{n=0}^{N}\frac{e^{i\pi(n+\frac{1}{2})x}-e^{i\pi((n-1)+\frac{1}{2})x}+e^{-i\pi((n-1)+\frac{1}{2})x}-e^{-i\pi(n+\frac{1}{2})x}}{e^{\frac{i\pi x}{2}}-e^{-\frac{i\pi x}{2}}}-1)=$$

$$=\frac{1}{2}(\sum_{n=0}^{N}\frac{\sin((n+\frac{1}{2})\pi x)-\sin(((n-1)+\frac{1}{2})\pi x)}{\sin(\frac{\pi x}{2})}-1)=\frac{1}{2}(\frac{\sin((N+\frac{1}{2})\pi x)-\sin(-\frac{1}{2}\pi x)}{\sin(\frac{\pi x}{2})}-1)=$$

$$=\frac{1}{2}(\frac{\sin((N+\frac{1}{2})\pi x)+\sin(\frac{1}{2}\pi x)}{\sin(\frac{\pi x}{2})}-1)=\frac{\sin((N+\frac{1}{2})\pi x)}{2\sin(\frac{\pi x}{2})}$$

$$\Rightarrow \delta(x)=\lim_{l\to 0}\frac{\sin(\frac{\pi x}{l})}{2\sin(\frac{\pi x}{2})}=\lim_{N\to\infty}\frac{\sin((N+\frac{1}{2})\pi x)}{2\sin(\frac{\pi x}{2})}=\lim_{N\to\infty}\frac{1}{2}\sum_{n=-N}^{N}e^{i\pi nx}$$

Aufgabe 30

$$\int_{h(x_1)}^{h(x_2)} dh(x)\, f(h(x))\delta(h(x))\overset{!}{=}\begin{Bmatrix} f(0) & x_1<x_0<x_2 \\ 0 & \text{sonst}\end{Bmatrix}=\int_{x_1}^{x_2} dx\, f(0)\delta(x_0-x)$$

$$dh(x)=h'(x)dx$$

$$f(0)=f(h(x_0))$$

$$\int_{h(x_1)}^{h(x_2)} dh(x)\, f(h(x))\delta(h(x))=\int_{x_1}^{x_2} dx\, f(h(x))h'(x)\delta(h(x))\overset{!}{=}\int_{x_1}^{x_2} dx\, f(h(x))\delta(x_0-x)$$

$\Rightarrow \delta(h(x))=\frac{\delta(x_0-x)}{h'(x)}=\frac{\delta(x_0-x)}{|h'(x_0)|}$ hier der Betrag weil Richtung egal (δ symm.),

$h'(x_0)$ statt $h'(x)$, weil nur an der Stelle x_0 interessant und dort identisch mit $h'(x)$

Aufgabe 31

$$(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2})\Phi(\vec{r},t)=0$$

$$\Delta\Phi(\vec{r},t)=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi(\vec{r},t)$$

$$\int_{\mathbb{R}^3}(\partial_x^2+\partial_y^2+\partial_z^2)\frac{\widetilde{\Phi(\vec{k},t)}}{(2\pi)^3}e^{i\vec{k}\vec{r}}d^3\vec{k}=\int_{\mathbb{R}^3}-\vec{k}^2\frac{\widetilde{\Phi(\vec{k},t)}}{(2\pi)^3}e^{i\vec{k}\vec{r}}d^3\vec{k}=\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}-\vec{k}^2\frac{\widetilde{\widetilde{\Phi(\vec{k},s)}}}{(2\pi)^4}e^{i(\vec{k}\vec{r}+st)}ds\, d^3\vec{k}$$

$$\int_{\mathbb{R}^3}\partial_t^2\frac{1}{c^2}\frac{\widetilde{\Phi(\vec{k},t)}}{(2\pi)^3}e^{i\vec{k}\vec{r}}d^3\vec{k}=\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\partial_t^2}{c^2}\frac{\widetilde{\widetilde{\Phi(\vec{k},s)}}}{(2\pi)^4}e^{i(\vec{k}\vec{r}+st)}ds\, d^3\vec{k}=\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}-s^2\frac{1}{c^2}\frac{\widetilde{\widetilde{\Phi(\vec{k},s)}}}{(2\pi)^4}e^{i(\vec{k}\vec{r}+st)}ds\, d^3\vec{k}$$

$\Rightarrow c^2\vec{k}^{\,2} = s^2 \;\Rightarrow\; s = \pm|\vec{k}\,|c$

nun gilt also: $\int\limits_{-\infty}^{\infty}\int\limits_{\mathbb{R}^3}\left(\frac{s^2}{c^2} - \vec{k}^{\,2}\right)\frac{\widetilde{\widetilde{\widetilde{\Phi(\vec{k}\,,s)}}}}{(2\pi)^4}e^{i(\vec{k}\vec{r}+st)}d\,s\,d^3\vec{r} = 0$

Das heißt, dass $\widetilde{\widetilde{\widetilde{\Phi(\vec{k}\,,s)}}}$ überall dort 0 sein muss, wo $s = \pm|\vec{k}\,|c$ nicht gilt.

$\Rightarrow \widetilde{\widetilde{\widetilde{\Phi(\vec{k}\,,s)}}} = a_1\delta(\frac{s}{c} - |\vec{k}\,|) + a_2\delta(\frac{s}{c} + |\vec{k}\,|)$

$\Rightarrow \Phi(\vec{r}\,,t) = \int\limits_{-\infty}^{\infty}\int\limits_{\mathbb{R}^3}\frac{a_1\delta(\frac{s}{c} - |\vec{k}\,|) + a_2\delta(\frac{s}{c} + |\vec{k}\,|)}{(2\pi)^4}e^{i(\vec{k}\vec{r}+st)}d\,s\,d^3\vec{k} =$

$= \int\limits_{-\infty}^{\infty}\int\limits_{\mathbb{R}^3}\frac{a_1\delta(\frac{s}{c} - |\vec{k}\,|)}{(2\pi)^4}e^{i(\vec{k}\vec{r}+st)}d\,s\,d^3\vec{k} + \int\limits_{-\infty}^{\infty}\int\limits_{\mathbb{R}^3}\frac{a_2\delta(\frac{s}{c} + |\vec{k}\,|)}{(2\pi)^4}e^{i(\vec{k}\vec{r}+st)}d\,s\,d^3\vec{k} =$

$= \frac{a_1}{(2\pi)^4}e^{i(\vec{k}\vec{r}+st)} + \frac{a_2}{(2\pi)^4}e^{i(\vec{k}\vec{r}-st)}$ mit $s = \pm|\vec{k}\,|c$