Einführung in die Theoretische Physik

Blatt 7

Aufgabe 25

$$\vec{V} = \begin{pmatrix} \frac{4}{3}x - 2y \\ 3y - x \\ 0 \end{pmatrix} \Rightarrow \mathrm{rot}(\vec{V}) = \begin{pmatrix} d_y 0 - d_z(3y - x) \\ d_z(\frac{4}{3}x - 2y) - d_x 0 \\ d_x(3y - x) - d_y(\frac{4}{3}x - 2y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\int_A d\vec{f}\,\mathrm{rot}(\vec{V}) = \int_A d\vec{f} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = A = \pi \cdot 3 \cdot 2 = 6\pi$$

$x = 3 \cdot \cos\varphi$

$y = 2 \cdot \sin\varphi$

$dx = -3\sin\varphi\, d\varphi$

$dy = 2\cos\varphi\, d\varphi$

$$\int_{\partial A} d\vec{r}\vec{V} = \int_{\partial A} V_x dx + V_y dy + V_z dz = \int_{\partial A} (\frac{4}{3}x - 2y)dx + (3y - x)dy + 0dz =$$

$$= \int_0^{2\pi} (4\cos\varphi - 4\sin\varphi)(-3\sin\varphi)d\varphi + (6\sin\varphi - 3\cos\varphi)(2\cos\varphi)d\varphi =$$

$$= \int_0^{2\pi} (-12\sin\varphi\cos\varphi + 12\sin^2\varphi + 12\sin\varphi\cos\varphi - 6\cos^2\varphi)d\varphi = \int_0^{2\pi} (12\sin^2\varphi - 6\cos^2\varphi)d\varphi =$$

$$= \int_0^{2\pi} \frac{12}{2}(1 + \sin(2\varphi)) - \frac{6}{2}(1 + \cos(2\varphi)) = \int_0^{2\pi} 6 + 6\sin(2\varphi)) - 3 + 3\cos(2\varphi)) =$$

$$= [3\varphi - 6\cos 2\varphi + 3\sin 2\varphi]_0^{2\pi} = [6\pi - 6 + 0 - 0 + 6 - 0] = 6\pi = \int_A d\vec{f}\,\mathrm{rot}(\vec{V})$$

$$* \quad \vec{V} = \begin{pmatrix} a\,x \\ b\,y \\ c\,z \end{pmatrix} \Rightarrow \mathrm{div}(\vec{V}) = a + b + c$$

$$\int_K \mathrm{dVol} \cdot \mathrm{div}(\vec{V}) = \mathrm{Vol} \cdot \mathrm{div}(\vec{V}) = \frac{4\pi}{3}R^3(a + b + c)$$

$x = r\cos\varphi\cos\theta$

$y = r\sin\varphi\cos\theta$

$z = r\sin\theta$

Tangenten Vektoren auf der Kugeloberfläche: $\vec{e}_\varphi$ and $\vec{e}_\theta$

$$\vec{t}_\varphi = \frac{\partial\vec{r}}{\partial\varphi} = \begin{pmatrix} -r\cos\theta\sin\varphi \\ r\cos\theta\cos\varphi \\ 0 \end{pmatrix}$$

$$\vec{t}_\theta = \frac{\partial\vec{r}}{\partial\theta} = \begin{pmatrix} -r\cos\varphi\sin\theta \\ -r\sin\varphi\sin\theta \\ r\cos\theta \end{pmatrix}$$

$$\Rightarrow \vec{n} = \vec{t}_\varphi \times \vec{t}_\theta = r^2 \begin{pmatrix} \cos^2\theta\cos\varphi \\ \cos^2\theta\sin\varphi \\ \sin^2\varphi\sin\theta\cos\theta + \cos^2\varphi\cos\theta\sin\theta \end{pmatrix} = r^2 \begin{pmatrix} \cos^2\theta\cos\varphi \\ \cos^2\theta\sin\varphi \\ \sin\theta\cos\theta \end{pmatrix}$$

$$\int\limits_{\partial K} d\,f\cdot\vec{n}\cdot\vec{V} = \int\limits_0^{2\pi} d\varphi \int\limits_{-\pi/2}^{\pi/2} d\theta\, R^2 \begin{pmatrix} \cos^2\theta\cos\varphi \\ \cos^2\theta\sin\varphi \\ \sin\theta\cos\theta \end{pmatrix} \cdot \begin{pmatrix} a\,R\cos\varphi\,\cos\theta \\ b\,R\sin\varphi\,\cos\theta \\ c\,R\sin\theta \end{pmatrix} =$$

$$= \int\limits_0^{2\pi} d\varphi \int\limits_{-\pi/2}^{\pi/2} d\theta\, R^3\ (\cos^2\theta\cos\varphi)(a\cos\varphi\,\cos\theta) +$$

$$+ (\cos^2\theta\sin\varphi)(b\sin\varphi\,\cos\theta) + (\sin\theta\cos\theta)(c\sin\theta) =$$

$$= \int\limits_0^{2\pi} d\varphi \int\limits_{-\pi/2}^{\pi/2} d\theta\, R^3\ (a\cos^3\theta\cos^2\varphi + b\cos^3\theta\sin^2\varphi + c\sin^2\theta\cos\theta =$$

mit $\int\limits_0^{2\pi} d\varphi\cos^2\varphi = \int\limits_0^{2\pi} d\varphi\sin^2\varphi = \pi$

$$= \int\limits_0^{2\pi} d\varphi\, R^3(a\,\pi\cos^3\theta + b\,\pi\,\cos^3\theta + c\,2\pi\,\cos\theta\sin^2\theta) =$$

with $\int d\theta\cos^3\theta = \frac{1}{3}\cos^2\theta\,\sin\theta + \frac{2}{3}\sin\theta,\ \int d\theta\cos\theta\sin^2\theta = \frac{1}{3}\sin^3\theta$

$$= R^3(\pi a[\tfrac{1}{3}\cos^2\theta\,\sin\theta + \tfrac{2}{3}\sin\theta]_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} + \pi b[\tfrac{1}{3}\cos^2\theta\,\sin\theta + \tfrac{2}{3}\sin\theta]_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} + 2c[\tfrac{1}{3}\sin^3\theta]_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} =$$

$$= \pi R^3(a\cdot 2\cdot\tfrac{2}{3} + b\cdot 2\cdot\tfrac{2}{3} + 2c\cdot 2\cdot\tfrac{1}{3}) = \tfrac{4\pi}{3}R^3(a+b+c) = \int\limits_K \mathrm{dVol}\cdot\mathrm{div}(\vec{V})$$

Aufgabe 26

$f_1(x) = A\exp(-a|x|), a > 0$

$$\tilde{f}_1(k) = \int\limits_{-\infty}^{+\infty} d\,x\, f_1(x) e^{-ikx} = A(\int\limits_0^{+\infty} d\,x\, e^{-(a+ik)x} + \int\limits_0^{+\infty} d\,x\, e^{-(a-ik)x}) =$$

$$= A(\left[\frac{e^{-(a+ik)x}}{-a-ik}\right]_0^\infty + \left[\frac{e^{-(a-ik)x}}{-a+ik}\right]_0^\infty) = A(\frac{1}{-a-ik} + \frac{1}{-a+ik}) = A\frac{-a+ik-a-ik}{a^2+k^2} = \frac{2A\,a}{a^2+k^2}$$

$$f_2(x) = \begin{cases} A & \text{für } -B < x < B \\ 0 & \text{sonst} \end{cases}$$

$$\tilde{f}_2(k) = \int\limits_{-\infty}^{+\infty} d\,x\, f_2(x) e^{-ikx} = \int\limits_{-B}^{+B} d\,x\, A\,e^{-ikx} = A\left[\frac{e^{-ikx}}{-i\,k\,x}\right]_{-B}^{+B} = \frac{A}{i\,k\,B}(e^{iBk} - e^{-iBk}) = \frac{2A}{i\,B}\sin(B\,k)$$

Aufgabe 27

a) Taylorentwicklung $\cos(x)$ um $x_0 = 0$:

$$\cos(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$

Taylorentwicklung $\cos(x^2)$ um $x_0 = 0$:

$$\cos(x) = \sum_{k=0}^{\infty} \frac{x^{4k}}{(2k)!}$$

b) $f_1(x+h) = \sum\limits_{k=0}^{\infty} \frac{\frac{d^k}{d\,x^k} f_1(x) h^k}{k!} = \sum\limits_{k=0}^{\infty} \frac{h^k \frac{d^k}{d\,x^k}}{k!} f_1(x) = e^{h\frac{d}{d\,x}} f_1(x)$

$f_2(x+h, y+j) = e^{h\frac{d}{d\,x}} f_2(x, y+j) = e^{j\frac{d}{d\,y}} e^{h\frac{d}{d\,x}} f_2(x, y)$