Einführung in die Theoretische Physik - Blatt 6

20.

a)

b) $\phi_1(x,y) = \ln(x + \sqrt{x^2+2})$

$$\mathrm{grad}\phi_1 = \begin{pmatrix} \frac{1 + \frac{2x}{2\sqrt{x^2+2}}}{x + \sqrt{x^2+2}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{x^2+2}+x}{(x+\sqrt{x^2+2})\sqrt{x^2+2}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{x^2+2}} \\ 0 \end{pmatrix}$$

$\phi_2(x,y) = 2x^2 + y^2$

$\mathrm{grad}\phi_2 = \begin{pmatrix} 4x \\ 2y \end{pmatrix}$

$\phi_3(x,y) = \cos(x+y)$

$\mathrm{grad}\phi_3 = \begin{pmatrix} -\sin(x+y) \\ -\sin(x+y) \end{pmatrix}$

22.

a) $\frac{1}{x^3}\frac{dx}{dt} = -\frac{a}{x^2} - 1$

$y := x^{-2} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-2}{x^3}$

$\frac{-1}{2}\frac{dy}{dt} = -ay - 1$

$\frac{dy}{dt} = 2ay + 2$

$\frac{dy}{2ay+2} = dt$

$\int_{y_0}^{y} \frac{dy'}{2ay'+2} = \int_{t_0}^{t} dt'$

$\left[\frac{1}{2a}\ln(2ay'+2)\right]_{y_0}^{y} = [t']_{t_0}^{t}$

$\frac{1}{2a}\ln\frac{2ay+2}{2ay_0+2} = t - t_0$

$\frac{1}{2a}(\ln(2ay+2) + C) = t$

$\ln(2ay+2) = 2at - C$

$2ay + 2 = e^{2at-C}$

$y = \frac{e^{2at-C} - 2}{2a}$

$x = \sqrt{\frac{2a}{e^{2at-C} - 2}}$

b) $\frac{dx}{dt} = a - bx^2$

$\frac{dx}{a - bx^2} = dt$

$\frac{1}{a - bx^2} = \frac{1}{(\sqrt{a} - \sqrt{b}x)(\sqrt{a} + \sqrt{b}x)}$

$\frac{1}{(\sqrt{a}+\sqrt{b}x)} + \frac{1}{(\sqrt{a}-\sqrt{b}x)} = \frac{(\sqrt{a}-\sqrt{b}x) + (\sqrt{a}+\sqrt{b}x)}{a - bx^2} = 2\sqrt{a}\frac{1}{a - bx^2}$

$$\int \frac{1}{a-b\,x^2} = \frac{1}{2\sqrt{a}}\int \left(\frac{1}{(\sqrt{a}+\sqrt{b}x)} + \frac{1}{(\sqrt{a}-\sqrt{b}x)}\right)d\,x = \frac{1}{2\sqrt{a}}\,\frac{1}{\sqrt{b}}\Big(\ln(\sqrt{a}+\sqrt{b}x) - \ln(\sqrt{a}-\sqrt{b}x)\Big) =$$

$$= \frac{1}{2\sqrt{a\,b}}\ln\frac{\sqrt{a}+\sqrt{b}\,x}{\sqrt{a}-\sqrt{b}\,x}$$

$$\int_{x_0}^{x} \frac{d\,x'}{a-b\,x'^2} = \int_{t_0}^{t} d\,t$$

$$\frac{1}{2\sqrt{a\,b}}\left[\ln\frac{\sqrt{a}+\sqrt{b}x'}{\sqrt{a}-\sqrt{b}x'}\right]_{x_0}^{x} = t - t_0$$

$$\frac{1}{2\sqrt{a\,b}}\left(\ln\frac{\sqrt{a}+\sqrt{b}x}{\sqrt{a}-\sqrt{b}x} + C'\right) = t$$

$$\frac{\sqrt{a}+\sqrt{b}x}{\sqrt{a}-\sqrt{b}x} = C\,e^{2\sqrt{a\,b}\,t}$$

$$\sqrt{a} + \sqrt{b}x = \sqrt{a}C\,e^{2\sqrt{a\,b}\,t} - \sqrt{b}x\,C\,e^{2\sqrt{a\,b}\,t}$$

$$\sqrt{b}x + \sqrt{b}x\,C\,e^{2\sqrt{a\,b}\,t} = \sqrt{a}C\,e^{2\sqrt{a\,b}\,t} - \sqrt{a}$$

$$x = \frac{\sqrt{a}(C\,e^{2\sqrt{a\,b}\,t} - 1)}{\sqrt{b}(C\,e^{2\sqrt{a\,b}\,t} + 1)}$$

23. $\vec{V}(x, y, z) = V_0(z - y, x, -x)$

a) $\vec{\nabla}\cdot\vec{V} = V_0\left(\frac{\partial(z-y)}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial(-x)}{\partial z}\right) = V_0 \cdot 0 = 0$

$\Rightarrow$ quellenfrei

b) $\vec{V} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = V_0\begin{pmatrix} z-y \\ x \\ -x \end{pmatrix} \overset{!}{=} \vec{\nabla}U$

$\Rightarrow \frac{\partial}{\partial x}U = v_1 = V_0(z-y) \Rightarrow U = V_0(z-y)x + C = V_0(-y\cdot x + z\cdot x + t_1(y,z))$

$\frac{\partial}{\partial y}U = v_2 = V_0(x) \Rightarrow U = V_0(x\cdot y) + C = V_0(x\cdot y + t_2(x,z))$

$\frac{\partial}{\partial z}U = v_3 = V_0(-x) \Rightarrow U = V_0(-x\cdot z) + C = V_0(-x\cdot z + t_3(x,y))$

$\Rightarrow$ there is no potential U such that $\vec{V} = \operatorname{grad} U$

c) $A = \int_\Gamma (K_x d\,x + K_y d\,y + K_z d\,z) = V_0$

i. $\Gamma = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + t\cdot\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$, $t = 0...1$

$x = 0,\ y = -1 + 3t,\ z = 1 - t$

$A = V_0\int_\Gamma (z-y)d\,x + x\,d\,y - x\,d\,z = V_0\int_0^1 (1 - t + 1 - 3t)0\,d\,t + 0\cdot 3d\,t - 0\cdot(-1)d\,t =$

$V_0[0]_0^1 = 0$

ii. $(0,-1,1) - (0,0,1)$: $x = 0,\ y = -1 + t, z = 1$

$\int_{\Gamma_1} (z-y)d\,x + x\,d\,y - x\,d\,z = \int_0^1 (2+t)0\,d\,t + 0\,d\,t - 0\,d\,t = 0$

$(0,0,1) - (0,0,0)$: $x = 0,\ y = 0, z = 1 - t$

$\int_{\Gamma_2} (z-y)dx + x\,dy - x\,dz = \int_0^1 (1-t)0\,dt + 0\,dt - 0\,dt = 0$

$(0,0,0) - (0,2,0)$: $x=0,\ y=2t,\ z=0$

$\int_{\Gamma_3} (z-y)dx + x\,dy - x\,dz = \int_0^1 (-2t)0\,dt + 0\,dt - 0\,dt = 0$

$\Rightarrow A = V_0(0+0+0) = 0$

iii. $\vec{r}(\zeta) = \begin{pmatrix} \sqrt{2}\sin\zeta \\ -\cos\zeta + \frac{\zeta}{\pi} \\ \cos\zeta + \frac{\zeta}{\pi} \end{pmatrix}$

P_1: $\sqrt{2}\sin\zeta = 0 \Rightarrow \zeta = k\pi, k \in \mathbb{Z}$

$-\cos\zeta + \frac{\zeta}{\pi} = -\cos(k\pi) + \frac{k\pi}{\pi} = k - \cos(k\pi) = -1 \Rightarrow k = 0$

$\cos\zeta + \frac{\zeta}{\pi} = \cos 0 + 0 = 1 \checkmark$

$\Rightarrow \zeta = 0$

P_2: $\sqrt{2}\sin\zeta = 0 \Rightarrow \zeta = k\pi, k \in \mathbb{Z}$

$-\cos\zeta + \frac{\zeta}{\pi} = -\cos(k\pi) + \frac{k\pi}{\pi} = k - \cos(k\pi) = 2 \Rightarrow k = 1$

$\cos\zeta + \frac{\zeta}{\pi} = \cos\pi + 1 = -1 + 1 = 0 \checkmark$

$\Rightarrow \zeta = \pi$

$x = \sqrt{2}\sin\zeta \Rightarrow dx = \sqrt{2}\cos\zeta d\zeta$

$y = -\cos\zeta + \frac{\zeta}{\pi} \Rightarrow dy = (\sin\zeta + \frac{1}{\pi})d\zeta$

$z = \cos\zeta + \frac{\zeta}{\pi} \Rightarrow dz = (-\sin\zeta + \frac{1}{\pi})d\zeta$

$A = V_0 \int_\Gamma (z-y)dx + x\,dy - x\,dz =$

$= V_0 \int_0^\pi (\cos\zeta + \frac{\zeta}{\pi} + \cos\zeta - \frac{\zeta}{\pi})\sqrt{2}\cos\zeta d\zeta + \sqrt{2}\sin\zeta(\sin\zeta + \frac{1}{\pi})d\zeta - \sqrt{2}\sin\zeta(\frac{1}{\pi} - \sin\zeta)d\zeta =$

$= V_0 \int_0^\pi (2\sqrt{2}\cos^2\zeta + 2\sqrt{2}\sin^2\zeta)d\zeta = V_0 \int_0^\pi 2\sqrt{2}d\zeta = 2\sqrt{2}V_0[\zeta]_0^\pi = 2\sqrt{2}\pi V_0$