Einführung in die Theoretische Physik

Blatt 10

Aufgabe 36

Halbkugelschale so, dass Schnittkante in der xy-Ebene, z-Achse=Symmetrieachse

Flächenladungsdichte: σ

Bestimmung des Potentials $\varphi(\vec{r})$ mit $\vec{r} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$

$$\varphi(\vec{r}) = \int_{\mathbb{R}^3} \frac{\rho}{\|\vec{r} - \vec{r}'\|} = \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} d\theta R^2 \cos\theta \frac{\sigma}{\| \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} - \begin{pmatrix} R\cos\varphi\cos\theta \\ R\sin\varphi\cos\theta \\ R\sin\theta \end{pmatrix} \|}$$

$$\| \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} - \begin{pmatrix} R\cos\varphi\cos\theta \\ R\sin\varphi\cos\theta \\ R\sin\theta \end{pmatrix} \| = R \| \begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ \sin\theta - \frac{z}{R} \end{pmatrix} \| = R\sqrt{\cos^2\varphi\cos^2\theta + \sin^2\varphi\cos^2\theta + \sin^2\theta - \frac{2\sin\theta\, z}{R} + \frac{z^2}{R^2}} =$$

$$= R\sqrt{1 - \frac{2\sin\theta\, z}{R} + \frac{z^2}{R^2}}$$

$$j := \sin\theta \Rightarrow d\theta\cos\theta = dj$$

$$\Rightarrow \varphi(\vec{r}) = \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} d\theta R^2 \cos\theta \frac{\sigma}{\| \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} - \begin{pmatrix} R\cos\varphi\cos\theta \\ R\sin\varphi\cos\theta \\ R\sin\theta \end{pmatrix} \|} = 2\pi R\sigma \int_0^1 dj \frac{1}{\sqrt{1 - 2j\frac{z}{R} + (\frac{z}{R})^2}}$$

$$\frac{\partial}{\partial j}\sqrt{1 - 2j\frac{z}{R} + (\frac{z}{R})^2} = \frac{-\frac{z}{R}}{\sqrt{1 - 2j\frac{z}{R} + (\frac{z}{R})^2}} \Rightarrow \int_0^1 dj \frac{1}{\sqrt{1 - 2j\frac{z}{R} + (\frac{z}{R})^2}} = -\frac{R}{z}\left[\sqrt{1 - 2j\frac{z}{R} + (\frac{z}{R})^2}\right]_0^1$$

$$\Rightarrow \varphi(\vec{r}) = -2\pi\frac{R^2}{z}\sigma\left[\sqrt{1 - 2j\frac{z}{R} + (\frac{z}{R})^2}\right]_0^1 = -2\pi\frac{R^2}{z}\sigma(\sqrt{1 - 2\frac{z}{R} + (\frac{z}{R})^2} - \sqrt{1 + (\frac{z}{R})^2}) =$$

$$= -2\pi\frac{R^2}{z}\sigma(|1 - \frac{z}{R}| - \sqrt{1 + (\frac{z}{R})^2})$$

sieht unschön aus, keine Ahnung obs richtig ist

Aufgabe 37

Paramterisierung der Oberfl. laut Angabe: $\vec{r}(\varphi, \psi) = \begin{pmatrix} (R + \rho\cos\psi)\cos\varphi \\ (R + \rho\cos\psi)\sin\varphi \\ \rho\sin\psi \end{pmatrix}$ $0 \leq \varphi < 2\pi,\, 0 \leq \psi < 2\pi$

Tangentenvekt. an die Oberfl.: $\vec{t}_1 = \frac{\partial\vec{r}}{\partial\varphi} = \begin{pmatrix} -(R + \rho\cos\psi)\sin\varphi \\ (R + \rho\cos\psi)\cos\varphi \\ 0 \end{pmatrix}$ $\vec{t}_2 = \frac{\partial\vec{r}}{\partial\psi} = \begin{pmatrix} -\rho\sin\psi\cos\varphi \\ -\rho\sin\psi\sin\varphi \\ \rho\cos\psi \end{pmatrix}$

$$\vec{t}_1 \times \vec{t}_2 = \rho\begin{pmatrix} (R + \rho\cos\psi)\cos\varphi\cos\psi - 0 \\ 0 + (R + \rho\cos\psi)\sin\varphi\cos\psi \\ (R + \rho\cos\psi)\sin^2\varphi\sin\psi + (R + \rho\cos\psi)\cos^2\varphi\sin\psi \end{pmatrix} = \rho\begin{pmatrix} (R + \rho\cos\psi)\cos\varphi\cos\psi \\ (R + \rho\cos\psi)\sin\varphi\cos\psi \\ (R + \rho\cos\psi)\sin\psi \end{pmatrix}$$

$$\|\vec{t}_1 \times \vec{t}_2\| = \rho(((R + \rho\cos\psi)\cos\varphi\cos\psi)^2 + ((R + \rho\cos\psi)\sin\varphi\cos\psi)^2 + ((R + \rho\cos\psi)\sin\psi)^2)^{\frac{1}{2}} =$$

$$= \rho(((R + \rho\cos\psi)\cos\psi)^2 + ((R + \rho\cos\psi)\sin\psi)^2)^{\frac{1}{2}} = \rho(R + \rho\cos\psi)$$

$$A = \int_0^{2\pi} d\varphi \int_0^{2\pi} d\psi \|\vec{t}_1 \times \vec{t}_2\| = 2\pi\rho\int_0^{2\pi} d\psi(R + \rho\cos\psi) = 2\pi\rho[R\psi + \rho\sin\psi]_0^{2\pi} = 2\pi\rho\, 2\pi R = 4\pi\rho R$$

Aufgabe 38

Aufgabe 39

a) $A = \int\limits_{P_1}^{P_2} d\,S = \int\limits_{x_1}^{x_2} d\,x \sqrt{1+y'^2}$

$f(y, y') = \sqrt{1+y'^2}$

ELG: $\frac{\partial f}{\partial y} - \frac{d}{d\,x}\frac{\partial f}{\partial y'} = 0$

$\Rightarrow 0 - \frac{d}{d\,x}\frac{y'}{\sqrt{1+y'^2}} = -\frac{y''\sqrt{1+y'^2} - \frac{y'y''}{\sqrt{1+y'^2}}}{1+y'^2} = \frac{y''(1+y'^2) - y'^2 y''}{(1+y'^2)^{\frac{3}{2}}} = \frac{y''}{(1+y'^2)^{\frac{3}{2}}} = 0$

$\Rightarrow y'' = 0 \Rightarrow y' = c_1 \Rightarrow y = c_1 x + c_2$

$y_1 = c_1 x_1 + c_2$

$y_2 = c_1 x_2 + c_2$

$\Rightarrow c_1 = \frac{y_2 - y_1}{x_2 - x_1}$, $c_2 = y_1 - \frac{y_2 - y_1}{\frac{x_2}{x_1} - 1} = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}$

b) $\frac{\partial f}{\partial y} - \frac{d}{d\,x}\frac{\partial f}{\partial y'} = 0 \Rightarrow \frac{\partial f}{\partial y} = \frac{d}{d\,x}\frac{\partial f}{\partial y'}$ (*)

$\frac{d}{d\,x}\left(f(y, y') - \frac{\partial f}{\partial y'} y'\right) = \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' - \frac{\partial f}{\partial y'} y'' - \frac{d}{d\,x}\frac{\partial f}{\partial y'}\, y' \overset{\text{nach (*)}}{=} \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y} y' = 0$

c) $A = \int\limits_{P_1}^{P_2} \frac{d\,S}{v} = \int\limits_{x_1}^{x_2} d\,x \frac{\sqrt{1+y'^2}}{\sqrt{2\,g\,y}}$

ELG: $\frac{\partial f}{\partial y} - \frac{d}{d\,x}\frac{\partial f}{\partial y'} = 0$

$\Rightarrow \frac{-g\sqrt{1+y'^2}}{\sqrt{2\,g\,y}^3} - \frac{d}{d\,x}\frac{y'}{\sqrt{1+y'^2}\sqrt{2\,g\,y}} = \frac{-g\sqrt{1+y'^2}}{\sqrt{2\,g\,y}^3} - \frac{\sqrt{1+y'^2}\sqrt{2\,g\,y}\,y'' - y'\frac{y'y''\sqrt{2\,g\,y}}{\sqrt{1+y'^2}} - y'\frac{g\,y'\sqrt{1+y'^2}}{\sqrt{2\,g\,y}}}{2\,g\,y\,(1+y'^2)} =$

$= \frac{-g(1+y'^2)^2 - 2\,g\,y\,y''(1+y'^2) + 2\,g\,y\,y''y'^2 + y'^2 g(1+y'^2)}{\sqrt{2\,g\,y}^3\sqrt{1+y'^2}^3} = \frac{-g - y'^2 g - 2\,g\,y\,y''}{\sqrt{2\,g\,y}^3\sqrt{1+y'^2}^3} \overset{!}{=} 0$

$\Rightarrow -y'^2 - 2y\,y'' - 1 = 0$

Test ob Zykloide eine Lösung dieser Gleichung darstellt:

$x = k(\lambda - \sin\lambda),\ y = k(1 - \cos\lambda)$

$\frac{d\,y}{d\,x} = \frac{d\,y}{d\,\lambda}\frac{d\lambda}{d\,x} = k\sin\lambda \frac{1}{k - k\cos\lambda} = \frac{\sin\lambda}{1 - \cos\lambda}$

$\frac{d^2 y}{d\,x^2} = \frac{d\frac{d\,y}{d\,x}}{d\lambda}\frac{d\lambda}{d\,x} = \frac{\cos\lambda\,(1-\cos\lambda) - \sin^2\lambda}{(1-\cos\lambda)^2}\,\frac{1}{k - k\cos\lambda} = \frac{\cos\lambda - 1}{k(1-\cos\lambda)^3} = \frac{-1}{k(1-\cos\lambda)^2}$

$-y'^2 - 2y\,y'' - 1 = \frac{-\sin^2\lambda}{(1-\cos\lambda)^2} - 2k(1-\cos\lambda)\frac{-1}{k(1-\cos\lambda)^2} - 1 =$

$= \frac{-1 + \cos^2\lambda + 2 - 2\cos\lambda}{(1-\cos\lambda)^2} - 1 = \frac{1 - 2\cos\lambda + \cos^2\lambda}{(1-\cos\lambda)^2} - 1 = 1 - 1 = 0$

Zykloide ist also Lösung