Einführung in die theoretische Physik - Blatt 1

1.

a)

i. $f_1(x) = a^x = e^{x\cdot\ln a}$

$\frac{\mathrm{df_1(x)}}{\mathrm{dx}} = e^{x\cdot\ln a}\cdot\ln a = a^x\cdot\ln a$

ii. $f_2(x) = x^x = e^{x\cdot\ln x}$

$\frac{\mathrm{df_2(x)}}{\mathrm{dx}} = e^{x\cdot\ln x}\cdot(1\cdot\ln x + x\cdot\frac{1}{x}) = x^x\,(\ln x + 1)$

iii. $f_3(x) = x^2\cdot\cos x^2$

$\frac{\mathrm{df_2(x)}}{\mathrm{dx}} = 2x\cdot\cos x^2 + x^2\cdot(-\sin x^2)\cdot 2x = 2x\cdot(\cos x^2 - x^2\sin x^2)$

b)

i. $g_1(x,y) = 4x^5 + 2x^2y + 9x\,y^3$

$\frac{\partial g_1}{\partial x} = 20x^4 + 4x\,y + 9y^3$

$\frac{\partial g_1}{\partial y} = 4x^5 + 2x^2 + 27x\,y^2$

ii. $g_2(x,y,z) = \frac{x}{\sqrt{x^2+y^2+z^2}}$

$$\frac{\partial g_2}{\partial x} = \frac{\sqrt{x^2+y^2+z^2} - x\cdot\frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2x}{x^2+y^2+z^2} = \frac{x^2+y^2+z^2-x^2}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$\frac{\partial g_2}{\partial y} = x\cdot\frac{\partial(x^2+y^2+z^2)^{-\frac{1}{2}}}{\partial y} = -x\cdot\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}\cdot 2y = \frac{-x\,y}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$\frac{\partial g_2}{\partial z} = x\cdot\frac{\partial(x^2+y^2+z^2)^{-\frac{1}{2}}}{\partial z} = -x\cdot\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}\cdot 2z = \frac{-x\,z}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

2.

a) $z\in\mathbb{C}\backslash\{0\}$:

$\frac{1}{z} = \frac{z^*}{z\cdot z^*} = \frac{z^*}{|z|^2} \Rightarrow \mathrm{Re}(\frac{1}{z}) = \frac{1}{|z|^2}\mathrm{Re}(z^*) = \frac{1}{|z|^2}\mathrm{Re}(z);\ \mathrm{Im}(\frac{1}{z}) = \frac{1}{|z|^2}\mathrm{Im}(z^*) = \frac{-1}{|z|^2}\mathrm{Im}(z)$

b) $z^5 = 1 \Rightarrow z_k = e^{\frac{2\pi k}{5}}, k\in\{0,1,2,3,4\}$

c) $-i(\sin\varphi - i\cos\varphi)^7 = i^7(-i\,e^{i\varphi})^7 = (-i^2)^7\,e^{7i\varphi} = e^{i(7\varphi)} = \cos 7\varphi + i\sin 7\varphi$

3.

a) $T\begin{pmatrix} t\\ x\\ y\\ z\end{pmatrix} = \begin{pmatrix} t'\\ x'\\ y'\\ z'\end{pmatrix} = \begin{pmatrix} t\\ x-v\,t\\ y\\ z\end{pmatrix}$

$\Rightarrow T = \begin{pmatrix} 1 & 0 & 0 & 0\\ -v & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$

b) Die Transformationen $T = \{\begin{pmatrix} 1 & 0 & 0 & 0\\ -v & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} | v\in\mathbb{R}\}$ bilden eine Gruppe, wenn:

i. für alle $T_1, T_2 \in T$: $T_1\cdot T_2\in T$

ii. für alle $T_1 \in T$ existiert $\mathbb{1}$ sodass $\mathbb{1} \cdot T_1 = T_1 \cdot \mathbb{1} = T_1$

iii. für alle $T_1 \in T$ existiert T_1^{-1} sodass $T_1 \cdot T_1^{-1} = T_1^{-1} \cdot T_1 = \mathbb{1}_4$

iv. für alle $T_1, T_2, T_3 \in T$: $(T_1 \cdot T_2) \cdot T_3 = T_1 \cdot (T_2 \cdot T_3)$

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v_2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; T_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

i. $$T_1 \cdot T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v_2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v_1 - v_2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

ii. $$\mathbb{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbb{1}_4 \cdot T_1 = T_1 \cdot \mathbb{1}_4 = T_1$$

iii. $$T_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ +v_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}:$$

$$T_1 \cdot T_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ +v_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v_1 + v_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{1}_4$$

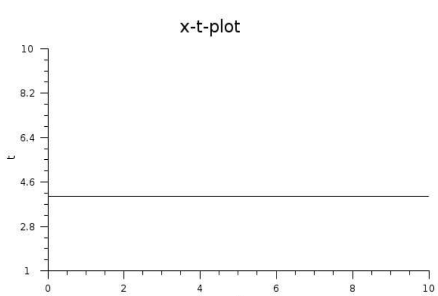
$$T_1^{-1} \cdot T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ +v_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ +v_1 - v_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{1}_4$$

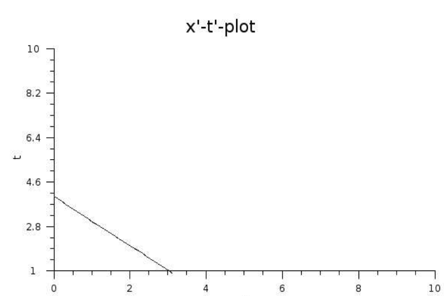
iv. $$(T_1 \cdot T_2) \cdot T_3 = (\begin{pmatrix} 1 & 0 & 0 & 0 \\ -v_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v_2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -v_1 - v_2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v_1 - v_2 - v_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_1 \cdot (T_2 \cdot T_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot (\begin{pmatrix} 1 & 0 & 0 & 0 \\ -v_2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}) =$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -v_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v_2 - v_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v_1 - v_2 - v_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (T_1 \cdot T_2) \cdot T_3$$

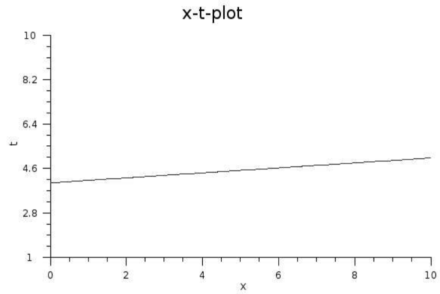
c) Achtung, bei Achsenbeschriftung ist x und t vertauscht!

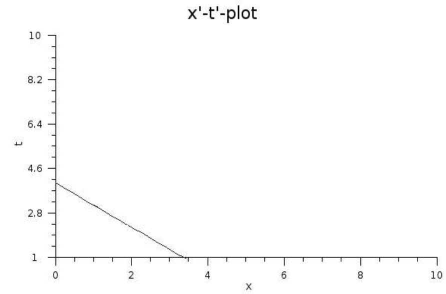
A: $\vec{r}(t) = (\Delta, 0, 0)$; $\vec{r}\,'(t) = (\Delta - \beta c\, t, 0, 0)$



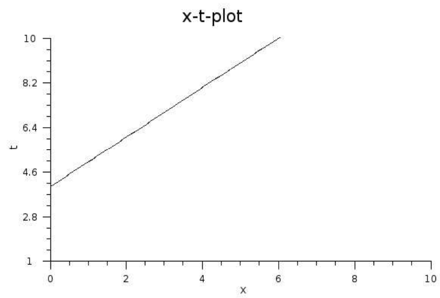


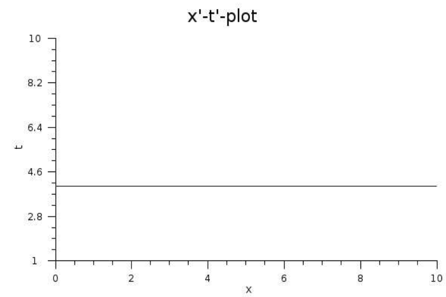
B: $\vec{r}\,(t) = (\Delta + 0.1\beta c\,t, 0, 0)$; $\vec{r}\,'(t) = (\Delta + 0.1\beta c\,t - \beta c\,t, 0, 0) = (\Delta - 0.9\beta c\,t, 0, 0)$





C: $\vec{r}\,(t) = (\Delta + \beta c\,t, 0, 0)$; $\vec{r}\,'(t) = (\Delta + \beta c\,t - \beta c\,t, 0, 0) = (\Delta, 0, 0)$





4. $\mathfrak{L}_v = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\beta = \frac{v}{c}$, $\gamma = \frac{1}{\sqrt{1 - (\frac{v}{c})^2}}$

a) $\|\vec{r}\,\| = \|(c\,t, x, y, z)\| = (c\,t)^2 - x^2 - y^2 - z^2$

$\|\vec{r}\,'\| = \|\vec{r}\,\mathfrak{L}_v\| = \|(\gamma c\,t - \gamma\beta x, \gamma x - \gamma\beta c\,t, y, z)\| =$

$= (\gamma c\,t)^2 - 2\gamma^2\beta c\,t\,x + (\gamma\beta x)^2 - (\gamma x)^2 + 2\gamma^2\beta c\,t\,x - (\gamma\beta c\,t)^2 - y^2 - z^2 =$

$= (\gamma^2 - \gamma^2\beta^2)(c\,t)^2 - (\gamma^2 - \gamma^2\beta^2)x^2 - y^2 - z^2 =$

$= (\frac{1 - \beta 2}{1 - \beta^2})(c\,t)^2 - (\frac{1 - \beta 2}{1 - \beta^2})x^2 - y^2 - z^2 = (c\,t)^2 - x^2 - y^2 - z^2 = \|(c\,t, x, y, z)\|$

b) $\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ 0 & \gamma(1 - \beta^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$\gamma = \frac{1}{\sqrt{1 - \beta^2}} \;\Rightarrow\; 1 - \beta^2 = \frac{1}{\gamma^2}$

$\begin{pmatrix} 1 & -\beta & 0 & 0 \\ 0 & \frac{1}{\gamma} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\gamma} & 0 & 0 & 0 \\ \beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\gamma} + \beta^2\gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$\frac{1}{\gamma} + \beta^2\gamma = \frac{1 + (\beta\gamma)^2}{\gamma} = \frac{1 + \frac{\beta^2}{1 - \beta^2}}{\gamma} = \frac{\frac{1}{1 - \beta^2}}{\gamma} = \frac{\gamma^2}{\gamma} = \gamma$

$\Rightarrow \mathfrak{L}_v^{-1} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

c)

d)

$$\gamma' = \gamma_1\gamma_2 + \gamma_1\gamma_2\beta_1\beta_2 = \gamma_1\gamma_2\,(1+\beta_1\beta_2)$$

$$-\,\beta\gamma = -\,\beta_2\gamma_2\gamma_1 - \beta_1\gamma_2\gamma_1$$

$$\Rightarrow \beta = \frac{\beta_2\gamma_2\gamma_1 + \beta_1\gamma_2\gamma_1}{\gamma_1\gamma_2\,(1+\beta_1\beta_2)} = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}$$

$$\Rightarrow v = \beta\,c = \frac{v_1+v_2}{1+\frac{v1\,v_2}{c^2}}$$

zur Kontrolle, z.z. $\gamma = \gamma'$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

$$\gamma_1\gamma_2\ (1+\beta_1\beta_2) = \frac{1}{\sqrt{1-\beta_1^2}}\frac{1}{\sqrt{1-\beta_2^2}}(1+\beta_1\beta_2) = \frac{1+\beta_1\beta_2}{\sqrt{(1-\beta_1^2)(1-\beta_2^2)}} = \frac{1}{\sqrt{\frac{(1-\beta_1^2)(1-\beta_2^2)}{(1+\beta_1\beta_2)^2}}} = \frac{1}{\sqrt{\frac{1+\beta_1^2\beta_2^2-\beta_1^2-\beta_2^2)}{(1+\beta_1\beta_2)^2}}}$$

$$= \frac{1}{\sqrt{\frac{(1+\beta_1\beta_2)^2}{(1+\beta_1\beta_2)^2} - \frac{\beta_1^2+2\beta_1 b_2+\beta_2^2}{(1+\beta_1\beta_2)^2}}} = \frac{1}{\sqrt{1-(\frac{\beta_1+\beta_2}{1+\beta_1\beta_2})^2}} = \frac{1}{\sqrt{1-\beta^2}}$$

$$\Rightarrow \mathfrak{L}_{v_1}\cdot\mathfrak{L}_{v_2} = \mathfrak{L}_{\frac{v_1+v_2}{1+\frac{v1\,v_2}{c^2}}}$$

e) $v \ll c \Rightarrow \gamma = \frac{1}{\sqrt{1-(\frac{v}{c})^2}} \approx 1$

$$\mathfrak{L}_v\vec{r} = \begin{pmatrix} \gamma c\,t - \gamma\beta x \\ \gamma x - \gamma\beta c\,t \\ y \\ z \end{pmatrix} \approx \begin{pmatrix} c\,t - \frac{v}{c}x \\ x - \frac{v}{c}c\,t \\ y \\ z \end{pmatrix} \approx \begin{pmatrix} c\,t \\ x - v\,t \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\,\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\vec{r}$$

$$\Rightarrow \mathfrak{L}_v = \begin{pmatrix} \gamma & -\,\gamma\beta & 0 & 0 \\ -\,\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\,\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$